Solutions:

Section 15.6

16. $M = \int_{-1}^{1}\int_{x^2}^{1} (y+1)\,dy\,dx = \int_{-1}^{1}(\frac{3}{2} - \frac{x^4}{2} - x^2)\,dx = \frac{32}{15}$

$M_x = \int_{-1}^{1}\int_{x^2}^{1} y(y+1)\,dy\,dx = \int_{-1}^{1}(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2})\,dx = \frac{48}{35}$

$M_y = \int_{-1}^{1}\int_{x^2}^{1} x(y+1)\,dy\,dx = \int_{-1}^{1}(\frac{3}{2}x - \frac{x^5}{2} - x^3)\,dx = 0$

$\Rightarrow \bar{x} = \frac{M_y}{M} = 0,\ \bar{y} = \frac{M_x}{M} = \frac{9}{14}$

$I_y = \int_{-1}^{1}\int_{x^2}^{1} x^2(y+1)\,dy\,dx = \int_{-1}^{1}(\frac{3}{2}x^2 - \frac{x^6}{2} - x^4)\,dx = \frac{16}{35}$

18. $M = \int_{0}^{20}\int_{-1}^{1} (1+\frac{x}{20})\,dy\,dx = \int_{0}^{20}(2+\frac{x}{10})\,dx = 60$

$M_x = \int_{0}^{20}\int_{-1}^{1} y(1+\frac{x}{20})\,dy\,dx = \int_{0}^{20} \left[(1+\frac{x}{20})(\frac{y^2}{2})\right]_{-1}^{1} dx = 0$

$M_y = \int_{0}^{20}\int_{-1}^{1} x(1+\frac{x}{20})\,dy\,dx = \int_{0}^{20}(2x+\frac{x^2}{10})\,dx = \frac{2000}{3}$

$\Rightarrow \bar{x} = \frac{100}{9},\ \bar{y} = 0$

$I_x = \int_{0}^{20}\int_{-1}^{1} y^2(1+\frac{x}{20})\,dy\,dx = \frac{2}{3}\int_{0}^{20}(1+\frac{x}{20})\,dx = 20$

20. $M = \int_{0}^{1}\int_{-y}^{y} (3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^3+2y)\,dy = \frac{3}{2}$

$M_x = \int_{0}^{1}\int_{-y}^{y} y(3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^4+2y^2)\,dy = \frac{16}{15}$

$M_y = \int_{0}^{1}\int_{-y}^{y} x(3x^2+1)\,dx\,dy = 0$

$\Rightarrow \bar{x} = 0,\ \bar{y} = \frac{32}{45}$

$I_x = \int_{0}^{1}\int_{-y}^{y} y^2(3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^5+2y^3)\,dy = \frac{5}{6}$

$I_y = \int_{0}^{1}\int_{-y}^{y} x^2(3x^2+1)\,dx\,dy = 2\int_{0}^{1}(\frac{3}{5}x^5+\frac{1}{3}y^3)\,dy = \frac{11}{30}$

$\Rightarrow I_0 = I_x + I_y = \frac{6}{5}$

28. $I_L = \int_{-2}^{2}\int_{-2}^{4}\int_{-1}^{(2-y)/2} [(x-4)^2+y^2]\,dz\,dy\,dx$ since the plane $y+2z=2$ is the top of the wedge

$$= \frac{1}{2}\int_{-2}^{2}\int_{-2}^{4}(x^2-8x+16+y^2)(4-y)\,dy\,dx$$

$$= \int_{-2}^{2}(9x^2-72x+162)\,dx = 696$$

$$M = \frac{1}{2}\times 3\times 6\times 4 = 36$$

30. (a) $M = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} xy(4-x^2)\,dy\,dx = \frac{k}{2}\int_0^2(4x^2-x^4)\,dx = \frac{32k}{15}$

(b) $M_{yz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kx^2y\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} x^2y(4-x^2)\,dy\,dx = \frac{k}{2}\int_0^2(4x^3-x^5)\,dx = \frac{8k}{3}$

$M_{xz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy^2\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} xy^2(4-x^2)\,dy\,dx = \frac{k}{3}\int_0^2(4x^{\frac{5}{2}}-x^{\frac{9}{2}})\,dx = \frac{256\sqrt{2}k}{231}$

$M_{xy} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxyz\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{x}} xy(4-x^2)^2\,dy\,dx = \frac{k}{4}\int_0^2(16x^2-8x^4+x^6)\,dx = \frac{256k}{105}$

$\Rightarrow \bar{x} = \frac{5}{4},\ \bar{y} = \frac{40\sqrt{2}}{77},\ \bar{z} = \frac{8}{7}$

34. Let $16-2x^2-2y^2 = 2x^2+2y^2$, we get $x^2+y^2=4 \Rightarrow$ The two parabolic surfaces intersect on $\{(x,y,z)\mid x^2+y^2=4,\ z=8\}$

$$\Rightarrow M = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2(x^2+y^2)}^{16-2(x^2+y^2)}\sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\sqrt{x^2+y^2}\,[16-4(x^2+y^2)]\,dy\,dx$$

$$= 4\int_0^{2\pi}\int_0^2 r(4-r^2)\,r\,dr\,d\theta = 4\int_0^{2\pi}\left(\frac{4}{3}r^3-\frac{1}{5}r^5\right)\Big|_0^2 d\theta = 4\int_0^{2\pi}\frac{64}{15}\,d\theta = \frac{512\pi}{15}$$

Additional Questions

1. $\left(\sum_{i=1}^{m} b_i\right)\left(\sum_{j=1}^{m} c_j\right) = \sum_{i=1}^{m}\left[b_i\left(\sum_{j=1}^{m} c_j\right)\right] = \sum_{i=1}^{m}\left[\sum_{j=1}^{m} b_i c_j\right] = \sum_{i=1}^{m}\sum_{j=1}^{m} b_i c_j$

2. $g(x) \overset{def}{:=} \sup_{y\in V} f(x,y)$. By definition, $\forall (x,y)\in U\times V,\ f(x,y) \le \sup_{(x,y)\in U\times V} f(x,y)$

$\Rightarrow$ For any fixed $x\in U$, ~~sup~~ $g(x) = \sup_{y\in V} f(x,y) \le \sup_{(x,y)\in U\times V} f(x,y)$

$\Rightarrow \sup_{x\in U} g(x) \le \sup_{(x,y)\in U\times V} f(x,y) \Rightarrow \sup_{x\in U}\sup_{y\in V} f(x,y) \le \sup_{(x,y)\in U\times V} f(x,y)$.

By definition, $\forall y \in V$, $f(x,y) \le g(x) = \sup_{y\in V} f(x,y)$ for any fixed $x \in U$

and $\forall x \in U$, $g(x) \le \sup_{x\in U} g(x) = \sup_{x\in U} \sup_{y\in V} f(x,y)$

$\Rightarrow \forall (x,y) \in U\times V$, $f(x,y) \le \sup_{x\in U}\sup_{y\in V} f(x,y)$

$\Rightarrow \sup_{(x,y)\in U\times V} f(x,y) \le \sup_{x\in U}\sup_{y\in V} f(x,y)$

$\Rightarrow \sup_{(x,y)\in U\times V} f(x,y) = \sup_{x\in U}\sup_{y\in V} f(x,y)$

$\Rightarrow \inf_{(x,y)\in U\times V} f(x,y) = -\sup_{(x,y)\in U\times V}[-f(x,y)] = -\sup_{x\in U}\sup_{y\in V}[-f(x,y)] = -\sup_{x\in U}[-\inf_{y\in V} f(x,y)]$

$= \inf_{x\in U}\inf_{y\in V} f(x,y)$

3. By definition, $\forall x \in U$, $f(x) \le \sup_{x\in U} f(x)$

$\Rightarrow \forall x \in V \subset U$, $f(x) \le \sup_{x\in U} f(x)$

$\Rightarrow \sup_{x\in V} f(x) \le \sup_{x\in U} f(x)$

$\Rightarrow \inf_{x\in V} f(x) = -\sup_{x\in V}[-f(x)] \ge -\sup_{x\in U}[-f(x)] = \inf_{x\in U} f(x)$

4. By definition, $\forall x \in U$, $f(x) \le \sup_{x\in U} f(x)$, $g(x) \le \sup_{x\in U} g(x)$

$\Rightarrow \forall x \in U$, $f(x)+g(x) \le \sup_{x\in U} f(x) + g(x) \le \sup_{x\in U} f(x) + \sup_{x\in U} g(x)$

$\Rightarrow \sup_{x\in U}(f(x)+g(x)) \le \sup_{x\in U} f(x) + \sup_{x\in U} g(x)$

$\Rightarrow \inf_{x\in U}(f(x)+g(x)) = -\sup_{x\in U}[-(f(x)+g(x))] \ge -\left(\sup_{x\in U}[-f(x)] + \sup_{x\in U}[-g(x)]\right)$

$= \inf_{x\in U} f(x) + \inf_{x\in U} g(x)$

5. By definition, $\forall x \in U$, $f(x) \le \sup_{x\in U} f(x) \Rightarrow \forall x\in U,\ cf(x) \le c \cdot \sup_{x\in U} f(x)$ since $c>0$

$$\Rightarrow \sup_{x\in U}[cf(x)] \le c\cdot \sup_{x\in U} f(x)$$

By definition, $\forall x\in U$, $cf(x) \le \sup_{x\in U}[cf(x)] \Rightarrow \forall x\in U,\ f(x) \le \frac{1}{c}\sup_{x\in U}[cf(x)]$ since $c>0$

$$\Rightarrow \sup_{x\in U} f(x) \le \frac{1}{c}\sup_{x\in U}[cf(x)] \Rightarrow c\sup_{x\in U} f(x) \le \sup_{x\in U}[cf(x)]$$

$$\Rightarrow c\sup_{x\in U} f(x) = \sup_{x\in U}[cf(x)]$$

$$\Rightarrow c\cdot \inf_{x\in U} f(x) = -c\sup_{x\in U}[-f(x)] = -\sup_{x\in U}[-cf(x)] = \inf_{x\in U}[cf(x)]$$

6. By definition, $\forall x\in U$, $f(x) \le \sup_{x\in U} f(x) \Rightarrow \forall x\in U,\ -f(x) \ge -\sup_{x\in U} f(x)$

$$\Rightarrow \inf_{x\in U}[-f(x)] \ge -\sup_{x\in U} f(x)$$

By definition, $\forall x\in U$, $-f(x) \ge \inf_{x\in U}[-f(x)] \Rightarrow \forall x\in U,\ f(x) \le -\inf_{x\in U}[-f(x)]$

$$\Rightarrow \sup_{x\in U} f(x) \le -\inf_{x\in U}[-f(x)] \Rightarrow -\sup_{x\in U} f(x) \ge \inf_{x\in U}[-f(x)]$$

$$\Rightarrow \inf_{x\in U}[-f(x)] = -\sup_{x\in U} f(x).$$